Topology Vol. 30, No. 3, pp. 429–438, 1991.
Printed in Great Britain

0040-9383/91 $3.00 + .00

EXAMPLES OF EXCEPTIONAL HOMOMORPHISMS WHICH HAVE NON-TRIVIAL EULER NUMBERS

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(*Received* 24 *January* 1990)

§0. INTRODUCTION

LET Σ be an oriented closed surface of genus ≥ 2. And let $G^r := \mathrm{Diff}^r_+(S^1)$ $(r = 0, 1, \ldots, \infty, \omega)$ be the group of all orientation preserving C^r-diffeomorphisms of S^1. Then there is a cohomology class $E \in H^2(G^r, \mathbb{Z})$ which is known as the Euler class. As Σ is $K(\pi_1(\Sigma), 1)$, there exists an isomorphism $H^2(\Sigma, \mathbb{Z}) \to H^2(\pi_1(\Sigma), \mathbb{Z})$ from the cohomology of spaces to the cohomology of abstract groups. This isomorphism there is the fundamental class $[\Sigma]$ of Σ in the cohomology group $H^2(\pi_1(\Sigma), \mathbb{Z})$. Then, for any homomorphism $\phi: \pi_1(\Sigma) \to G^r$, we define the Euler number $\mathrm{eu}(\phi)$ of ϕ by

$$\mathrm{eu}(\phi) = \langle \phi^*(E), [\Sigma] \rangle.$$

As is known, for such ϕ, the following inequality (called Milnor-Wood inequality) holds:

$$|\mathrm{eu}(\phi)| \leq |\chi(\Sigma)|,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ ([5, 7]).

Recently Ghys considered the influence of the qualitative properties of ϕ on the Euler number $\mathrm{eu}(\phi)$ of ϕ in his paper [1]. To describe this work we must first explain the classification of the minimal sets of ϕ.

$\pi_1(\Sigma)$ acts on S^1 through ϕ (this action is simply called ϕ-action on S^1). A non-empty, closed, ϕ-invariant subset A of S^1 is called *minimal*, if it is minimal with these properties. A minimal set of ϕ exists and is one of the following for each ϕ. That is:

(F) A finite orbit,
(D) S^1,
(EMS) A Cantor set $C \subset S^1$ such that the ϕ-orbit of each point of C is dense in C.

As is well known, if ϕ has the type (F) then $\mathrm{eu}(\phi) = 0$. And for any Σ and any integer n with $|n| \leq |\chi(\Sigma)|$, there exists a homomorphism $\phi: \pi_1(\Sigma) \to \mathrm{PSL}(2, \mathbb{R})$ such that $\mathrm{eu}(\phi) = n$. For such ϕ, it has type (D) if $n \neq 0$ (see the following theorem). And it is easy to see that there exists a homomorphism $\phi: \pi_1(\Sigma) \to \mathrm{SO}(2)$ with the property (D) and $\mathrm{eu}(\phi) = 0$. Then the condition (D) has no influence on the Euler number. On the other hand, as for homomorphisms with the property (EMS), Sacksteder [6] constructed a homomorphism $\phi: \pi_1(\Sigma_2) \to G^\infty$ with the property (EMS) whose Euler number $\mathrm{eu}(\phi) = 0$, where Σ_2 is the closed oriented surface of genus 2.

Recently Ghys obtained the following result.

THEOREM 1. ([1]) *For any homomorphism*

$$\phi: \pi_1(\Sigma) \to \mathrm{Diff}^r_+(S^1) \ (r = 2, 3, \ldots, \infty, \omega)$$

of type (EMS), *we have*

(1) $|\mathrm{eu}(\phi)| \leq |\chi(\Sigma)| - 1$ *if* $r \geq 2$,
(2) $\mathrm{eu}(\phi) = 0$ *if* $r = \omega$.

From this theorem, we take a great interest in the existence of a homomorphism $\phi:\pi_1(\Sigma) \to G^r$ $(2 \leq r \leq \infty)$ with the property (EMS) and of $\mathrm{eu}(\phi) \neq 0$. Ghys and Sergiescu have recently constructed an example of a homomorphism $\phi:\pi_1(\Sigma_{12}) \to G^\infty$ with the property (EMS) which has non-trivial Euler number $\mathrm{eu}(\phi) = 1$. But we feel this example a little modest, because the inequality (1) of Theorem 1 suggests that there might exist a homomorphism $\phi:\pi_1(\Sigma_{12}) \to G^\infty$ with $\mathrm{eu}(\phi) = |\chi(\Sigma_{12})| - 1 = 21$. But so far no one knows other examples or constructions of such ϕ. This gives us the following question:

Question. Is the Ghys inequality best possible? Here we call the inequality (1) in Theorem 1, the Ghys inequality.

The purpose of this paper is to prove the following theorem which is a partial answer to the Question.

Main Theorem. (1) *For any integer* m *with* $|m| \leq 1$, *there exists a homomorphism* $\phi:\pi_1(\Sigma_2) \to \mathrm{Diff}_+^\infty(S^1)$ *with the property* (EMS) *such that*

$$\mathrm{eu}(\phi) = m.$$

(2) *For any* Σ (*genus* ≥ 3) *and for any integer* m *of* $|m| \leq |\chi(\Sigma)| - 2$, *there exists a homomorphism* $\phi:\pi_1(\Sigma) \to \mathrm{Diff}_+^\infty(S^1)$ *with the property* (EMS) *such that*

$$\mathrm{eu}(\phi) = m.$$

Main Theorem (1) guarantees the best possibility of the Ghys inequality in case of genus $(\Sigma) = 2$. By the Main Theorem (2), the following problem remains;

Problem. Does there exist an exceptional homomorphism $\phi:\pi_1(\Sigma) \to \mathrm{Diff}_+^r(S^1)$ (genus $(\Sigma) \geq 3$ and $r = 2, 3, \ldots, \infty$) such that

$$\mathrm{eu}(\phi) = |\chi(\Sigma)| - 1 \quad ?$$

In §1, we recall the algorithm of Milnor to compute Euler number $\mathrm{eu}(\phi)$ of any homomorphism $\phi:\pi_1(\Sigma) \to G^r$.

In §2, we show that for any group G, any commutator $[f, g] \in G$ and any integer $n \geq 1$, $[f, g]^{2n-1}$ is represented as a product of n commutators in G. This plays an important role for the proof of the main theorem (see §4).

In §3, we define a group $\mathrm{PL}_+^{k;n}(S^1) \subset G^0$. And we prove the following two facts;

(1) There exists an exceptional homomorphism

$$\phi: \mathrm{PL}_+^{k;n}(S^1) \to G^\infty.$$

(2) $[\mathrm{PL}_+^{k;n}(S^1), \mathrm{PL}_+^{k;n}(S^1)] \cap \mathrm{SO}(2) \neq \phi$ and $\mathrm{PL}_+^{k;n}(S^1)$ contains sufficiently many elements to prove the Main Theorem (2).

In §4, we prove the main theorem.

§1. THE ALGORITHM OF MILNOR

We recall the Milnor's algorithm for computing the Euler number $\mathrm{eu}(\phi)$ for a homomorphism

$$\phi:\pi_1(\Sigma_k) \to G^0.$$

The universal covering group $\tilde{G}^0$ of G^0 is identified with the group of homeomorphisms f of $\mathbb{R}$ such that $f \circ T_1 = T_1 \circ f$. And we put that ρ: $\tilde{G}^0 \to G^0$ is the covering projection. On Σ_k, choose a meridian-longitude system $\{\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k\}$. We always take the system respecting the orientation of Σ_k. That is, at the point of intersection, the direction α_i followed by that of β_i gives the orientation of Σ_k. As is well known,

$$\pi_1(\Sigma_k) = \langle \alpha_1, \beta_1, \ldots, \alpha_k, \beta_k \mid [\alpha_1, \beta_1] \ldots [\alpha_k, \beta_k] = 1 \rangle.$$

For the homomorphism ϕ, take arbitrary lifts $\overline{\phi(\alpha_i)}, \overline{\phi(\beta_i)} \in \tilde{G}^0$ of $\phi(\alpha_i), \phi(\beta_i) \in G^0$ by ρ. Then $[\overline{\phi(\alpha_i)}, \overline{\phi(\beta_i)}]$ is independent of a choice of the lifts of $\phi(\alpha_i)$ and $\phi(\beta_i)$. Now,

$$[\overline{\phi(\alpha_1)}, \overline{\phi(\beta_1)}] \ldots [\overline{\phi(\alpha_k)}, \overline{\phi(\beta_k)}]$$

covers the identity of G^0. Thus it is of the form $T_m (m \in \mathbb{Z})$. This number m turns out to be the Euler number $\mathrm{eu}(\phi)$. See [5] for more details.

For a subgroup H of G^0 and a homomorphism

$$\phi: \pi_1(\Sigma_k) \to H,$$

$\mathrm{eu}(\phi)$ denotes the Euler number of the composite homomorphism

$$\phi: \pi_1(\Sigma_k) \to H \xrightarrow{i} G^0,$$

where i is the natural inclusion.

For two homomorphisms $\phi_i: \pi_1(\Sigma_{k_i}) \to G^0 (\mathrm{i} = 1, 2)$, define a homomorphism

$$\phi_1 \# \phi_2: \pi_1(\Sigma_{k_1 + k_2}) \to G^0,$$

by using the connected sum of Σ_1 and Σ_2. Then it is easy to see that

$$\mathrm{eu}(\phi_1 \# \phi_2) = \mathrm{eu}(\phi_1) + \mathrm{eu}(\phi_2).$$

For any subgroup H of G^0 we define homomorphisms

$$c_k: \pi_1(\Sigma_k) \to H (k = 1, 2, \ldots)$$

by

$$c_k(\gamma) = \mathrm{id}_{S^1} \text{ for any } \gamma \in \pi_1(\Sigma_k).$$

§2. THE COVERING SPACES OF A PUNCTURED TORUS AND REPRESENTATIONS BY COMMUTATORS

Let $\tilde{F}$, F be compact connected orientable surfaces with boundary, and let $\tilde{r}$ and r be (respectively) the number of their boundary components of $\partial\tilde{F}$, ∂F. The following result is due to Massey, Heim and Stöcker.

THEOREM 2.1 ([4], [3]) *Let $\tilde{F}$, F and $\tilde{r}$, r be as above. For any $n \in \mathbb{N}$, there exists an n-fold covering map $\pi: \tilde{F} \to F$ if $\chi(\tilde{F}) = n\chi(F)$ and $r \leq \tilde{r} \leq nr$.*

From this theorem, we have the following proposition about a representation by a product of commutators.

COROLLARY 2.2. *Let G be a group. Given two elements f, $g \in G$, we put $h = [f, g] = fgf^{-1}g^{-1}$. Then for any $k \geq 1$, h^{2k-1} is represented as a product of k commutators in G.*

Proof. Let Σ_k' be an orientable compact surface of genus $k \geq 1$, with the boundary $\partial\Sigma_k' = S^1$. Then there exists a $(2k - 1)$-fold covering map $\rho_k: \Sigma_k' \to \Sigma_1'$ by the Theorem 2.1.

$\pi_1(\Sigma'_j)$, the fundamental group of Σ'_j ($j \geq 1$), is a free group of rank $2j$. We may choose the generators

$$\alpha_i, \beta_i \in \pi_1(\Sigma'_k)\ (i = 1, 2, \ldots, k),\ \alpha, \beta \in \pi_1(\Sigma'_1)$$

possessing the property:

$$(*) \quad (\rho_k)_*(\partial_k) = \partial,$$

where $\partial_k = [\alpha_1, \beta_1] \ldots [\alpha_k, \beta_k]$ and $\partial = [\alpha, \beta]$. Define a homomorphism $H: \pi_1(\Sigma') \to G$ by $H(\alpha) = f$, $H(\beta) = g$, and put:

$$f_i = H \circ (\rho_k)_*(\alpha_i), \quad g_i = H \circ (\rho_k)_*(\beta_i), \quad (\mathrm{i} = 1, 2, \ldots, k).$$

Then we have:

$$\begin{aligned}[f_1, g_1] \circ \ldots \circ [f_k, g_k] &= H \circ (\rho_k)_*([\alpha_1, \beta_1] \ldots [\alpha_k, \beta_k]) \\ &= H \circ (\rho_k)_*(\partial_k) \\ &= H(\partial^{2k-1}) \quad \text{by } (*) \\ &= h^{2k-1}.\end{aligned}$$

This completes the proof. □

§3. DEFINITIONS AND PROPERTIES OF $\mathrm{PL}_+^{k;n}(S^1)$

In this section, we first define the group $\mathrm{PL}_+^{k;n}(S^1) \subset G^0$ which plays an important role in proving the main theorem, and discuss its properties.

For each integer $k \geq 2$, let $\mathbb{Q}_k$ be the ring of k-adic numbers; $p \cdot k^q (p, q \in \mathbb{Z})$, and let $\mathrm{GA}(\mathbb{Q}_k)$ be the group of affine homeomorphisms of type $(k^n, p \cdot k^q)(x) = k^n \cdot x + p \cdot k^q$ $(n, p, q \in \mathbb{Z})$. And $\mathrm{PL}_+^k(\mathbb{R})$ denotes the group of piecewise affine homeomorphisms h of $\mathbb{R}$ which have following property:

There exist $\{x_m\}_{m \in Z} \subset \mathbb{Q}_k$ and $\{\gamma_m\}_{m \in Z} \subset \mathrm{GA}(\mathbb{Q}_k)$ such that

(1) $\{x_m\}_{m \in Z} \subset \mathbb{R}$ has no accumulation point.
(2) $h|_{[x_m, x_{m+1}]} = \gamma_m|_{[x_m, x_{m+1}]}$.

Then, for each integer $n \geq 1$, $\widetilde{C_n \mathrm{PL}_+^k}(S^1)$ denotes the subgroup of $\mathrm{PL}_+^k(\mathbb{R})$ each element of which commutes with the translation T_n. In particular, we denote $\widetilde{C_1 \mathrm{PL}_+^k}(S^1)$ by $\widetilde{\mathrm{PL}}_+^k(S^1)$. Before the definition of $\mathrm{PL}_+^{k;n}(S^1)$, we prepare the following lemma.

LEMMA 3.1. *Define M_a by $M_a(x) = ax$, $a, x \in \mathbb{R}$ For any $a, b \in \mathbb{R}$, $a \neq 0$, we have that*
(1) $M_a^{-1} = M_{1/a}$,
(2) $M_a \circ T_b = T_{ab} \circ M_a$.

Proof. (1) $M_a \circ M_{1/a}(x) = a((1/a)x) = x = (1/a)(ax) = M_a \circ M_{1/a}(x)$ for any $x \in \mathbb{R}$. This means that $M_a^{-1} = M_{1/a}$.

(2) $M_a \circ T_b(x) = a(x + b) = ax + ab = T_{ab} \circ M_a(x)$ for any $x \in \mathbb{R}$. Then we have that $M_a \circ T_b = T_{ab} \circ M_a$. □

Now let $\widetilde{\mathrm{PL}}_+^{k;n}(S^1)$ be the subgroup of $\tilde{G}^0$ which is conjugate to $\widetilde{C_n \mathrm{PL}_+^k}(S^1)$ by a multiplication map M_n^{-1}. That is

$$\widetilde{\mathrm{PL}}_+^{k;n}(S^1) = M_n^{-1} \circ \widetilde{C_n \mathrm{PL}_+^k}(S^1) \circ M_n.$$

Then by using the formulas $M_n^{-1} = M_{1/n}$ and $M_n \circ T_1 = T_n \circ M_n$ in Lemma 3.1, we can show that $\widetilde{\mathrm{PL}}^{k;n}_{+}(S^1) \subset \tilde{G}^0$. Let $\mathrm{PL}^{k;n}_{+}(S^1)$ denote the group of orientation preserving homeomorphisms of $S^1 = \mathbb{R}/\mathbb{Z}$ induced from $\widetilde{\mathrm{PL}}^{k;n}_{+}(S^1)$. And we note that $\mathrm{PL}^{k;1}_{+}(S^1) = \mathrm{PL}^{k}_{+}(S^1)$. In the rest of this section, we investigate the properties of these groups defined above. Let us start from the following lemma.

Lemma 3.2. *For each integer $m \geq 1$, we have that*

$$M_m^{-1} \circ \widetilde{\mathrm{PL}}^{k;n}_{+}(S^1) \circ M_m \subset \widetilde{\mathrm{PL}}^{k;mn}_{+}(S^1).$$

Proof. By the definition of $\widetilde{\mathrm{PL}}^{k;n}_{+}(S^1)$, each element in it is of the form $M_n^{-1} \circ f \circ M_n (f \in \widetilde{C_n \mathrm{PL}^{k;n}_{+}}(S^1))$. Then we have that $M_m^{-1} \circ (M_n^{-1} \circ f \circ M_n) \circ M_m = M_{mn}^{-1} \circ f \circ M_{mn}$ belongs to $\widetilde{\mathrm{PL}}^{k;mn}_{+}(S^1)$. Since f is arbitrary, this completes the proof. □

Lemma 3.3. *For any integer $m \geq 1$, we have that*

$$\widetilde{\mathrm{PL}}^{k;n}_{+}(S^1) \subset \widetilde{\mathrm{PL}}^{k;mn}_{+}(S^1).$$

Proof.

$$\begin{aligned}
\widetilde{\mathrm{PL}}^{k;n}_{+}(S^1) &= M_n^{-1} \circ \widetilde{C_n \mathrm{PL}^{k}_{+}}(S^1) \circ M_n \\
&= M_n^{-1} \circ M_m^{-1} \circ M_m \circ \widetilde{C_n \mathrm{PL}^{k}_{+}}(S^1) \circ M_m^{-1} \circ M_m \circ M_n \\
&\subset M_{mn}^{-1} \circ \widetilde{C_{mn} \mathrm{PL}^{k}_{+}}(S^1) \circ M_{mn} \\
&= \widetilde{\mathrm{PL}}^{k;mn}_{+}(S^1).
\end{aligned}$$

□

The next lemma is one of the main tools to prove the main theorem.

Theorem 3.4. *For any integer $k \geq 2$, there exist $f, g \in \widetilde{C_{n_k} \mathrm{PL}^{k}_{+}}(S^1) (n_k = (k^2 + k + 1)(k^3 + 1))$ such that*

$$[f, g] = T_{(k^2-1)(k^3-1)}.$$

Proof. Define $f, g' \in \widetilde{C_{n_k} \mathrm{PL}^{k}_{+}}(S^1)$ as follows:

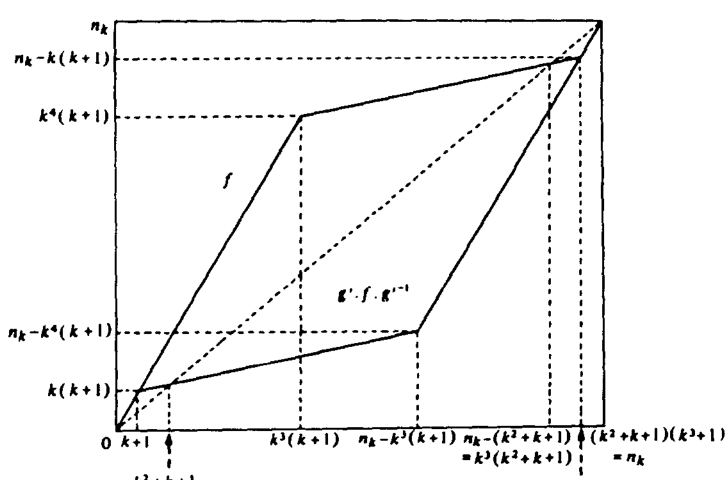

Fig. 1.

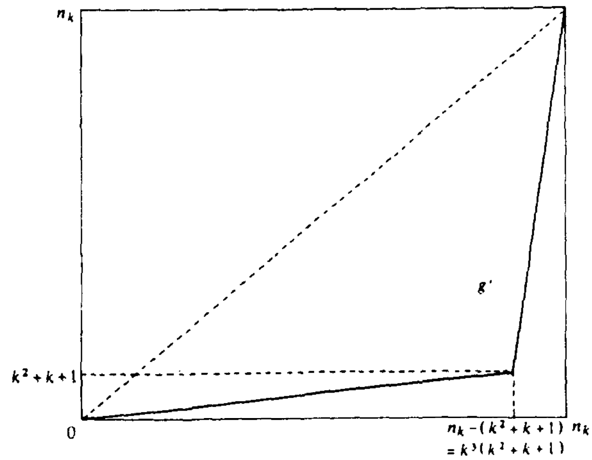

Fig. 2.

From these graphs, it follows that

$$\begin{aligned} f &= T_{(k^4-k)(k+1)} \circ g' \circ f \circ g'^{-1} \circ T^{-1}_{(k^3-1)(k+1)} \\ &= T_{(k^4-k-k^3+1)(k+1)} \circ T_{(k^3-1)(k+1)} \circ g' \circ f \circ (T_{(k^3-1)(k+1)} \circ g')^{-1}. \end{aligned}$$

And putting $g = T_{(k^3-1)(k+1)} \circ g'$, then we have

$$\begin{aligned} [f, g] &= T_{(k^4-k-k^3+1)(k+1)} \\ &= T_{(k^3-1)(k-1)(k+1)} \\ &= T_{(k^3-1)(k^2-1)}. \end{aligned}$$

This completes the proof. $\square$

Take any positive integer k and put $n = n_k$, and moreover take any $f, g \in \widetilde{C_n \mathrm{PL}^k_+}(S^1)$ as in Theorem 3.4, and fix them all. If we put $f_k = M_n^{-1} \circ f \circ M_n$ and $g_k = M_n^{-1} \circ g \circ M_n$, then we have that f_k and g_k belong to the group $\widetilde{\mathrm{PL}}^{k;n}_+(S^1)$ by definition. Using the Lemma 3.1.(2), we have the following equalities by straightforward calculation:

$$[f_k, g_k] = T_{(k^2-1)(k^3-1)/n_k},$$

$$\frac{(k^2-1)(k^3-1)}{n_k} = \frac{(k-1)^2}{k^2-k+1}.$$

Putting $m_k = k^2 - k + 1$, we have the following corollary:

Corollary 3.5. *For each integer $k \geq 2$, there exist $f, g \in \widetilde{\mathrm{PL}}^{k;n_k}_+(S^1)$ such that*

$$[f, g] = T_{(m_k-k)/m_k} = T_{(k-1)^2/m_k},$$

where $n_k = (k^2 + k + 1)(k^3 + 1)$.

Definition. (1) Let H be a subgroup of $\mathrm{Diff}^0_+(\mathbb{R})$ (resp. G^0). Then H is called *exceptional* if and only if the natural action of H on $\mathbb{R}$ (resp. S^1) has no closed orbit and has an orbit O_x of some $x \in \mathbb{R}$ (resp. $x \in S^1$) which is not dense in $\mathbb{R}$ (resp. S^1).

(2) Let Γ be any group. For any homomorphism $\phi: \Gamma \to \mathrm{Diff}^0_+(\mathbb{R})$ (resp. $\phi: \Gamma \to G^0$), ϕ is called *exceptional* if and only if the image group $\mathrm{Im}(\phi) = \phi(\Gamma)$ is exceptional.

To return to the subject, in [2] Ghys and Sergiescu constructed an exceptional homomorphism $\theta\colon \mathrm{PL}^2_+(\mathbb{R}) \to \mathrm{Diff}^\infty(\mathbb{R})$ which has the following property:

$$(*) \quad \theta(T_n) = T_n \text{ for any } n \in \mathbb{Z}.$$

By using this property θ naturally induces an exceptional homomorphism $\theta'\colon \widetilde{\mathrm{PL}}^2_+(S^1) \to \tilde{G}^\infty$. And moreover, θ' induces an exceptional homomorphism $\varphi\colon \mathrm{PL}^2_+(S^1) \to G^\infty$. By a little check of [2], we can see that this construction on the dyadic version is valid for k-adic version. Then we have the following:

THEOREM 3.6. (see [2]) *For each integer $k \geq 2$, there exists an exceptional homomorphism $\theta\colon \mathrm{PL}^k_+(\mathbb{R}) \to \mathrm{Diff}^\infty_+(\mathbb{R})$ such that*

$$\theta(T_n) = T_n \textit{ for any } n \in \mathbb{Z}.$$

This theorem leads to the following corollary:

COROLLARY 3.7. *For any positive integer k, n $(k \geq 2)$, there exists an exceptional homomorphism $\varphi\colon \mathrm{PL}^{k;n}_+(S^1) \to \mathrm{Diff}^\infty_+(S^1)$ which preserves the Euler number. That is, for any homomorphism $\phi\colon \pi_1(\Sigma) \to \mathrm{PL}^{k;n}_+(S^1)$, it holds that*

$$\mathrm{eu}(\varphi \circ \phi) = \mathrm{eu}(\phi).$$

Proof. Define a homomorphism $\tilde{\varphi}\colon \widetilde{\mathrm{PL}}^{k;n}_+(S^1) \to \mathrm{Diff}^\infty_+(\mathbb{R})$ by using θ in the Theorem 3.6 as follows: for any $f \in \widetilde{\mathrm{PL}}^{k;n}_+(S^1)$,

$$\tilde{\varphi}(f) = M_n^{-1} \circ \theta(M_n \circ f \circ M_n^{-1}) \circ M_n.$$

Since θ preserves translations T_n by $n \in \mathbb{Z}$, then we have, for any $f \in \widetilde{\mathrm{PL}}^{k;n}_+(S^1)$,

$$\begin{aligned}
\tilde{\varphi}(f) \circ T_1 &= M_n^{-1} \circ \theta(M_n \circ f \circ M_n^{-1}) \circ T_n \circ M_n \\
&= M_n^{-1} \circ \theta(M_n \circ f \circ M_n^{-1} \circ T_n) \circ M_n \\
&= M_n^{-1} \circ \theta(T_n \circ M_n \circ f \circ M_n^{-1}) \circ M_n \\
&= M_n^{-1} \circ T_n \circ \theta(M_n \circ f \circ M_n^{-1}) \circ M_n \\
&= T_1 \circ \tilde{\varphi}(f).
\end{aligned}$$

This means that $\mathrm{Im}(\tilde{\varphi}) \subset \widetilde{\mathrm{Diff}}^\infty_+(S^1)$. And for any translation T_m by $m \in \mathbb{Z}$, a straightforward calculation yields:

$$\tilde{\varphi}:(T_m) = T_m.$$

Then we define a homomorphism $\varphi\colon \mathrm{PL}^{k;n}_+(S^1) \to \mathrm{Diff}^\infty_+(S^1)$ so that the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{\mathrm{PL}}^{k;n}_+(S^1) & \xrightarrow{\tilde{\varphi}} & \widetilde{\mathrm{Diff}}^\infty_+(S^1) \\
\Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \rho} \\
\mathrm{PL}^{k;n}_+(S^1) & \xrightarrow{\varphi} & \mathrm{Diff}^\infty_+(S^1)
\end{array}$$

Next we show this φ preserves Euler numbers. For any closed oriented surface Σ (genus $g \geq 2$), choose generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ of the fundamental group $\pi_1(\Sigma)$ of Σ as in §1. Then for any homomorphism $\phi : \pi_1(\Sigma) \to \mathrm{PL}_+^{k;n}(S^1)$ $(\mathrm{eu}(\phi) = m)$, we have that

$$\begin{aligned}\mathrm{eu}(\varphi \circ \phi) &= [\tilde{\varphi}(\overline{\phi(\alpha_1)}), \tilde{\varphi}(\overline{\phi(\beta_1)})] \circ \ldots \circ [\tilde{\varphi}(\overline{\phi(\alpha_g)}), \tilde{\varphi}(\overline{\phi(\beta_g)})](0)\\ &= \tilde{\varphi}[\overline{\phi(\alpha_1)}, \overline{\phi(\beta_1)}] \ldots [\overline{\phi(\alpha_g)}, \overline{\phi(\beta_g)}](0)\\ &= \tilde{\varphi}(T_m)(0)\\ &= T_m(0) = m = \mathrm{eu}(\phi).\end{aligned}$$

Then φ preserves the Euler numbers, and this completes the proof. □

§4. PROOF OF THE MAIN THEOREM

To prove the main theorem, we give several lemmas on how many commutators in $\widetilde{\mathrm{PL}}_+^{k;n}(S^1)$ we need to represent a translation $T_n (n \in \mathbb{Z})$ as a product.

LEMMA 4.1. *$T_\varepsilon(\varepsilon = 1, -1)$ is represented as a product of 2 commutators in $\widetilde{\mathrm{PL}}_+^{2;63}(S^1)$*

Proof. By Corollary 3.5, there exist $f, g \in \widetilde{\mathrm{PL}}_+^{2;63}(S^1)$ such that $[f, g] = T_{1/3}$. Then by using Corollary 2.2, $[f, g]^3$ is represented as a product of 2 commutators in $\widetilde{\mathrm{PL}}_+^{2;63}(S^1)$. Now $T_{-1} = T_1^{-1}$, then T_{-1} is also. This completes the proof. □

From now throughout this section, we use the notations as follows:

k: positive even integers,
$n_k := (k^2 + k + 1)(k^3 + 1)$,
$m_k := k^2 - k + 1$.

LEMMA 4.2. *T_{k-1} is represented as a product of $(k/2) + 2$ commutators in $\widetilde{\mathrm{PL}}_+^{k;(k-1)n_k}(S^1)$.*

Proof. By Corollary 3.5, there exist $f, g \in \widetilde{\mathrm{PL}}_+^{k;n_k}(S^1)$ such that $[f, g] = T_{(m_k - k)/m_k} = T_{(k-1)^2/m_k}$. Then it is easy to see by straightforward calculations that

$$\frac{m_k - k}{m_k}k + \frac{m_k - k}{m_k}\frac{1}{k-1} = k - 1.$$

This yields that

$$T^{k-1}_{(m_k - k)/m_k} \circ T_{(m_k - k)/m_k} \circ T_{(k-1)/m_k} = T_{k-1}.$$

By Lemma 3.3, f and g belong to $\widetilde{\mathrm{PL}}_+^{k;(k-1)n_k}(S^1)$. Then $T_{(m_k - k)/m_k}$ and $T^{k-1}_{(m_k - k)/m_k}$ are represented as a product of 1 and $k/2$ commutators in $\widetilde{\mathrm{PL}}_+^{k;(k-1)m_k}(S^1)$ by Corollary 2.2. And by Lemma 3.2, $M_{k-1}^{-1} \circ f \circ M_{k-1}$ and $M_{k-1}^{-1} \circ g \circ M_{k-1}$ belong to $\mathrm{PL}_+^{k;(k-1)n_k}(S^1)$, and it is easy to see that

$$[M_{k-1}^{-1} \circ f \circ M_{k-1}, M_{k-1}^{-1} \circ g \circ M_{k-1}] = T_{(k-1)/m_n}.$$

Then T_{k-1} is represented as a product of $(k/2) + 1 + 1$ commutators in $\widetilde{\mathrm{PL}}_+^{k;(k-1)n_k}(S^1)$. And this completes the proof. □

LEMMA 4.3. *T_{k-2} is represented as a product of $(k/2)+1$ commutators in $\widetilde{\mathrm{PL}}_+^{k,(k-1)^2 n_k}(S^1)$.*

Proof. Using Corollary 3.5, there exist $f, g \in \widetilde{\mathrm{PL}}_+^{k,n_k}(S^1)$ such that $[f,g] = T_{(m_k-k)/m_k}$. Then we have the following equality by an easy calculation:

$$\frac{m_k - k}{m_k}(k-1) - \frac{1}{m_k} = k - 2$$

This means that

$$T^{k-1}_{(m_k-k)/m_k} \circ T^{-1}_{1/m_k} = T_{k-2}.$$

Since $f, g \in \widetilde{\mathrm{PL}}_+^{k,n_k}(S^1) \subset \widetilde{\mathrm{PL}}_+^{k,(k-1)^2 n_k}(S^1)$, then we have by Corollary 2.2, that $T^{k-1}_{(m_k-k)/m_k}$ is represented as a product of $k/2$ commutators in $\widetilde{\mathrm{PL}}_+^{k,(k-1)^2 n_k}(S^1)$. Since $m_k - k = (k-1)^2$ and we have $M^{-1}_{(k-1)^2} \circ g \circ M_{(k-1)^2}, M^{-1}_{(k-1)^2} \circ M_{(k-1)^2} \in \widetilde{\mathrm{PL}}_+^{k,(k-1)^2 n_k}(S^1)$, the commutators of these elements in this order is T_{1/m_k}. Then T_{k-2} is represented as a product of $(k/2)+1$ commutators in $\widetilde{\mathrm{PL}}_+^{k,(k-1)^2 n_k}(S^1)$. This completes the proof. □

Now let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be a generator system of the fundamental group $\pi_1(\Sigma)$ (genus$(\Sigma) = g$) as in §1. Let $\rho: \tilde{G}^0 \to G^0$ be the projection.

THEOREM 4.4. (1) There exists a homomorphism $\phi_\varepsilon: \pi_1(\Sigma_2) \to \mathrm{PL}_+^{2,63}(S^1)$ ($\varepsilon = 1, -1$) such that

$$\mathrm{eu}(\phi) = \varepsilon.$$

(2) For any Σ(genus ≥ 3) and for any integer m of $|m| \leq |\chi(\Sigma)| - 2$, there exists a positive even integer k and a homomorphism $\phi: \pi_1(\Sigma) \to \mathrm{PL}_+^{k,n_k}(S^1)$ such that

$$\mathrm{eu}(\phi) = m,$$

where $n_k = (k^2 + k + 1)(k^3 + 1)$.

Proof. (1) By Lemma 4.1., there exist $f_i \in \widetilde{\mathrm{PL}}_+^{2,63}(S^1)$ ($i = 1,2,3,4$) such that $[f_1, f_2][f_3, f_4] = T_\varepsilon$. We can define a homomorphism

$$\phi: \pi_1(\Sigma_2) \to \mathrm{PL}_+^{2,63}(S^1)$$

by $\phi(\alpha_i) = \rho(f_{2i-1})$ and $\phi(\beta_i) = \rho(f_{2i})$ ($i = 1,2$). Then the Euler number $\mathrm{eu}(\phi)$ is

$$\begin{aligned}\mathrm{eu}(\phi) &= [\phi(\alpha_1), \phi(\beta_1)](\phi(\alpha_2), \phi(\beta_2)](0) \\ &= [f_1, f_2][f_3, f_4](0) \\ &= \varepsilon\end{aligned}$$

completing the proof of (1).

(2) *Step* 1. We prove that for any $\Sigma_g (g \geq 3)$, there exists a homomorphism $\phi_g: \pi_1(\Sigma_g) \to \mathrm{PL}_+^{k,(k-1)n_k}(S^1)$ ($k = 2g-4$) such that $\mathrm{eu}(\phi) = 2g - 5$.

Put that $k = 2g - 4$. Since $g \geq 3$, k is positive even integer. Then by Lemma 4.2, $T_{k-1} = T_{2g-5}$ is represented as a product of $(k/2) + 2 = g$ commutators in $\widetilde{\mathrm{PL}}_+^{k,(k-1)n_k}(S^1)$. Then using the same method of the proof of (1), we can construct a homomorphism $\phi_g: \pi_1(\Sigma_g) \to \mathrm{PL}_+^{k,(k-1)n_k}(S^1)$ such that $\mathrm{eu}(\phi_g) = 2g - 5$. This completes the proof of Step 1.

Step 2. We prove that for any $\Sigma_g (g \geq 3)$, there exists a homomorphism $\psi_g: \pi_1(\Sigma_g) \to \mathrm{PL}_+^{k,(k-1)^2 n_k}(S^1)$ ($k = 2g - 2$) such that $\mathrm{eu}(\psi_g) = 2g - 4$.

Let $k = 2g - 2$. Since $g \geq 3$ then k is positive even integer. Then by Lemma 4.3, $T_{k-2} = T_{2g-4}$ is represented as a product of $(k/2) + 1$ commutators in $\widetilde{\mathrm{PL}}_+^{k,(k-1)^2 n_k}(S^1)$. As in the construction of the homomorphism ϕ in the proof of (1), we can construct a homomorphism $\psi_g:\pi_1(\Sigma_g)\to \mathrm{PL}_+^{k,(k-1)^2n_k}(S^1)$ such that $\mathrm{eu}(\psi_g)=2g-4$. This completes Step 2.

Step 3. [Completion of the proof of (2)]

Let c_g be as in §1 (i.e. the trivial homomorphism). For any integer n with $|n|\le|\chi(\Sigma_g)|=2g-4$, by changing the orientation of Σ_g, one can assume that $n\ge 0$. If $n=0$, the trivial homomorphism c_g clearly satisfies the required conditions. It is sufficient to prove the result for $n\ge 1$.

If n is a positive even integer, then $3\le(n+4)/2\le g$. By Step 2, the Euler number of a connected sum homomorphism $\psi_{(n+4)/2}\# c_{g-(n+4)/2}:\pi_1(\Sigma_g)\to \mathrm{PL}_+^{k,(k-1)^2n_k}(S^1)$ $(k=n+2)$ is equal to $\mathrm{eu}(\phi_{(n+4)/2})=n$.

If n is a positive odd integer, then $3\le(n+5)/2\le g$. By Step 1, the Euler number of a connected sum homomorphism $\phi_{(n+5)/2}\# c_{g-(n+5)/2}:\pi_1(\Sigma_g)\to \mathrm{PL}_+^{k,(k-1)^2n_k}(S^1)$ $(k=n+1)$ is equal to $\mathrm{eu}(\phi_{(n+5)/2})=n$. This completes the Step 3. And this completes the proof of the Theorem 4.4. □

Proof of Main Theorem. For any Σ (genus ≥ 2), the examples of Sacksteder give us exceptional homomorphisms $\phi:\pi_1(\Sigma)\to G^\infty$ with Euler number $\mathrm{eu}(\phi)=0$. Then it is sufficient to prove the case of $m\ne 1$. But Theorem 4.4 and Corollary 3.7 give us the required exceptional homomorphism $\phi:\pi_1(\Sigma)\to G^\infty$ whose Euler number $\mathrm{eu}(\phi)=m$. This completes the proof. □

Acknowledgements—The author wishes to thank Professors H. Suzuki, T. Nishimori and M. Takamura for helpful comments during the preparation of this paper.

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